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States on Effect Algebras That Have the ϕ-Symmetry Property

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The relationship between the property of having a full set of states and the archimedean property in the case of a ϕ-symmetric effect algebra is explored, and equivalent conditions are obtained.

1. INTRODUCTION

An *effect algebra* is an algebraic system $(E, 0, u, \oplus)$ consisting of a set E, a partially defined binary operation $\oplus$, together with the special element 0 called the *zero* and the special element u called the *unit*, with $0 \neq u$, satisfying the following axioms for all $a, b, c \in E$.

(i) [Commutativity] if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(ii) [Associativity] If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(iii) [Orthosupplementation] For every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = u$. We denote this element b by a'.

(iv) [Zero–Unit Law] $u \oplus a$ is defined if and only if $a = 0$.

The prototypical example that motivates the study of effect algebras is the set $\mathcal{E}(\mathcal{H})$ of all self-adjoint operators T on a Hilbert space $\mathcal{H}$ such that $O \leq T \leq I$, where O is the zero operator and I is the identity operator. In this case $S \oplus T$ is defined if and only if $S + T \leq I$.

Let E, F be two effect algebras with units u, v, respectively. A map ψ: $E \mapsto F$ is called a *homomorphism* if (i) $\psi(u) = v$, and (ii) for $a, b \in E$ such

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that $a \oplus b$ exists, $\psi(a) \oplus \psi(b)$ exists in F and $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$. A one-to-one effect algebra homomorphism from E onto F is an *isomorphism* if $\psi(a) \oplus \psi(b)$ exists in F implies $a \oplus b$ exists in E.

Effect algebras were first introduced by Foulis and Bennett (1974). We refer the reader to this work for a thorough discussion of their properties. We shall assume familiarity with these ideas on the reader's part. We shall also assume that the reader is familiar with the concepts of partially ordered abelian groups. Goodearl (1986) may be consulted for details.

Given a partially ordered abelian group G and an element e in its positive cone G^+, we define $G^+[0, e] := \{g \in G\text{: } 0 \leq g \leq e\}$. It is readily verified that $G^+ [0, e]$ is an effect algebra under the definition $a \oplus b\text{: } = a + b$ if and only if $a + b \leq e$.

An effect algebra E is called an *interval effect algebra* if it is isomorphic to an effect algebra of the form $G^+ [0, e]$, where G is a partially ordered abelian group. We refer the reader to Bennett and Foulis (n.d.), where the study of the class of interval effect algebras was first launched.

A natural partial order can be introduced in an effect algebra by defining $a \leq c$ in E if and only if there exists a $b \in E$ such that $a \oplus b = c$. An effect algebra E has the *Riesz decomposition property* if $a \leq b \oplus c$ in E implies that there exists, in E, $b_1 \leq b$ and $c_1 \leq c$ such that $a = b_1 \oplus c_1$. If the partial order in an effect algebra E turns out to be a lattice order, we say that E is lattice-ordered. A lattice-ordered effect algebra E is said to have the *ϕ-symmetry property* if, for $a, b \in E$, $a \wedge b = 0$ implies that $a \oplus b$ exists. It is easy to verify that a lattice-ordered effect algebra that has the Riesz decomposition property is ϕ-symmetric (Ravindran, 1996). The concept of effect algebras that have the ϕ-symmetry property was first introduced by Bennett and Foulis (1995). A study of effect algebras that have the Riesz decomposition property was launched in Ravindran (1996). During the IQSA Conference in Berlin in 1996, the author was asked if a ϕ-symmetric effect algebra can be viewed as an MV-algebra [see Chang (1958) for an introduction to MV-algebras] with appropriate definitions. The answer is yes. Also, ϕ-symmetric effect algebras are lexicomorphically equivalent to Boolean D-posets (Kopka and Chovanec, 1996). Since many of the results for ϕ-symmetric effect algebras follow from the results for effect algebras with Riesz decomposition property (Ravindran, 1996), the same is true for MV-algebras and Boolean D-posets. A forthcoming paper from the author explores these connections in detail.

2. THE UNIVERSAL GROUP

The following fundamental result is proved in Bennett and Foulis (n.d.).

Theorem 2.1. Given an interval effect algebra E, there exists a partially ordered abelian group G, a generating cone G^+, and a nonzero element $u \in G^+$ such that the interval $G^+[0, u]$ generates G^+ and the following conditions are satisfied:

(i) E is isomorphic to the interval effect algebra $G^+[0, u]$.

(ii) Every additive map $\phi: E \mapsto K$ [that is, $\phi(a \oplus b) = \phi(a) + \phi(b)$] can be extended uniquely to a group homomorphism $\phi^*: G \mapsto K$. This group G is unique up to isomorphism.

The group G in the above theorem is called the *universal group* for E. It is easy to verify (Bennett and Foulis, n.d.) that the element u in the above theorem is an order unit for G. In the case of $\mathscr{E}(\mathscr{H})$, the universal group is the additive group of all self-adjoint operators on $\mathscr{H}$ and the positive cone is the collection of all positive self-adjoint operators on $\mathscr{H}$.

In Ravindran (1996) the universal group of Theorem 2.1 in the case of an effect algebra E with the Riesz decomposition property is explicitly constructed, and the result for effect algebras that have the φ-symmetry property follows as a corollary. The key elements of this construction, originally due to Baer (1949) (in his study of free sums of abelian groups), and later by Wyler (1966) (in his study of clans), are outlined below.

The first step is to form *words* $(a_1, \ldots, a_n)$, where $a_i \in E$ and $n \in \mathbb{N}$. The *length* of a word $(a_1, \ldots, a_n)$ is defined to be n. For each $a \in E$ one may define the word (a) of length 1 and one may abuse the notation by writing simply a instead of (a). The notation $|W|$ is used to denote the length of a word W. If $W = (a_1, \ldots, a_n)$, the a_i are referred to as the *components* of W. Two words are equal if their lengths are equal and the corresponding components are equal.

Next introduce a binary operation $+$ on the collection $\mathscr{W}(E)$ of all words as follows: for two words $W_1 := (a_1, \ldots, a_n)$, $W_2 := (b_1, \ldots, b_m)$, the word $W_1 + W_2 := (a_1, \ldots, a_n, b_1, \ldots, b_m)$. It is readily verified that $\mathscr{W}(E)$ is a semigroup (with no identity).

Consider a word $W_1 = (a_1, \ldots, a_n)$. A new word W_2 can be obtained from W_1 with $|W_2| = |W_1| - 1$ using the following rule: if $a_k \oplus a_{k+1}$ exists for some k, the pair a_k, a_{k+1} is replaced by the element $a_k \oplus a_{k+1}$. Denote this transition by writing $W_1 \to W_2$, and call this rule an *elementary transformation.* In Baer's terminology W_1 and W_2 are *directly similar.*

For two words W^1, W^2, define $W^1 \sim W^2$ if $\exists$ a finite number of words

$$W_0 := W^1, W_1, W_2, \ldots, W_{m-1}, W_m := W^2 \qquad \text{with} \quad m \geqq 1$$

and satisfying the following: for $0 \leqq i \leqq m - 1$, we have either $W_i = W_{i+1}$, or $W_i \to W_{i+1}$, or $W_{i+1} \to W_i$. In this case, W^1 and W^2 are said to be related by an *elementary chain* of *length m.*

Observe that $\sim$ is an equivalence relation on $\mathcal{W}(E)$, so we can form the collection of all equivalence classes $\mathcal{G}^+$. Define $+$ in $\mathcal{G}^+$ by $[W_1] + [W_2] := [W_1 + W_2]$, for $W_1, W_2 \in \mathcal{W}(E)$. It is easy to verify that $\mathcal{G}^+$ is the quotient semigroup $\mathcal{W}(E)/\sim$ consisting of all the equivalence classes with the binary operation $+$. Since $\mathcal{W}(E)$ is associative, so is $\mathcal{G}^+$. Given $(a_1, \ldots, a_n) \in \mathcal{W}(E)$, we have $(a_1, \ldots, a_n) = (a_1) + \ldots + (a_n)$ in $\mathcal{W}(E)$. Consequently $[(a_1, \ldots, a_n)] = [a_1] + \ldots + [a_n]$ in $\mathcal{G}^+$.

It has been proved in Ravindran (1996) that, in the case of an effect algebra that has the Riesz decomposition property, the partially ordered abelian group generated by $\mathcal{G}^+$ is the universal group for E. Moreover, G has the Riesz interpolation property. As a consequence, when E has the ϕ-symmetry property, G is lattice-ordered, and we have the following theorem.

Theorem 2.2. (a) Let E be an effect algebra that has the ϕ-symmetry property. Then there exists an order unit u in the universal group $\mathcal{G}$ for E such that E is isomorphic to the interval effect algebra $\mathcal{G}^+[0, u]$, and $\mathcal{G}$ is lattice-ordered.

(b)Conversely, if $\mathcal{G}$ is a lattice-ordered abelian group with order unit u, then $\mathcal{G}^+ [0, u]$ is an interval effect algebra that has the ϕ-symmetry property and $\mathcal{G}$ is its universal group.

It should be noted that part of the above theorem may be viewed as a special case of Wyler's results for clans. In fact, one can easily show that much of the terminology of Boolean D-posets (Kôpka and Chovanec, 1996) and generalized D-posets (Hedlíková and Pulmannová, 1995) can be recast in terms of Wyler's clans.

3. STATES AND THE ARCHIMEDEAN PROPERTY

The unit interval [0, 1] is an effect algebra under the natural definition $a \oplus b := a + b$ if and only if $a + b \leqq 1$. It is an interval effect algebra and its universal group is $\mathbb{R}$.

Definition 3.1. Let E be an effect algebra with unit u. A *state* s on E is an effect algebra homomorphism from E to [0, 1]. The set of all states on E will be denoted by $\mathcal{S}(E)$. A nonempty subset M of $\mathcal{S}(E)$ is called *full* if $s(a) \leq s(b)$ for all $s \in M$ implies that $a \leq b$. A *state* on a partially ordered abelian group with order unit u is an additive map t: $G \mapsto \mathbb{R}$ such that $t(u) = 1$ and $t(G^+) \subseteq \mathbb{R}^+$.

Following Goodearl (1986), denote the collection of all states on G by $S(G, u)$. Any state $t \in S(G, u)$, when restricted to $E := G^+[0, u]$, is a state on the interval effect algebra E. On the other hand, any state $s \in \mathcal{S}(E)$ extends to a state $s^* \in S(G, u)$ by Theorem 2.1. Thus $\mathcal{E}(E)$ is full if and only if $S(G, u)$ determines the order in E. Regarding states on interval effect algebras and other related details, the reader may consult Bennett and Foulis (n.d.).

Proposition 3.2. Let E be an effect algebra with a full set of states M. For $a \in E$, define the evaluation map $\hat{a}$: M $\mapsto [0, 1]$ by $\hat{a}(s) = s(a)$. Let $\hat{E}^M = \{\hat{a}\text{:a} \in E\}$. Define, for $\hat{a}, \hat{b} \in \hat{E}^M$, $\hat{a} \oplus \hat{b} := \hat{a} + \hat{b}$ if and only if $\hat{a} + \hat{b} \leqq \hat{u}$. Then $\hat{E}^M$ is an effect algebra with unit $\hat{u}$ and zero $\hat{0}$, and E is isomorphic to $\hat{E}^M$.

Proof. First note that if $a \oplus b$ exists in E, then $(\widehat{a \oplus b})(s) = s(a \oplus b) = s(a) \oplus s(b)$ for all states $s \in M$. Thus $(\widehat{a \oplus b}) = \hat{a} \oplus \hat{b}$. The fact that $\hat{E}^M$ is an effect algebra requires only a routine verification of the axioms of an effect algebra. Since $(\widehat{a \oplus b}) = \hat{a} \oplus \hat{b}$, $f(a \oplus b) = f(a) \oplus f(b)$. The map f is one-to-one since $a \leq b$ in E if and only if $s(a) \leq s(b)$ for all $s \in M$. Also, by the definition of $\hat{E}^M$, we conclude that f is onto and $f(u) = \hat{u}$. Finally, suppose $\hat{a} \oplus \hat{b} = f(a) \oplus f(b)$ exists. Then $s(a) + s(b) \leq 1 = s(u)$ for all $s \in M$. This implies that $s(a) \leq s(b')$ for all $s \in M$ and, since M is full, we conclude that $a \leq b'$ in E. This implies that (Foulis and Bennett, 1994) $a \oplus b$ exists in E.

Let E be an effect algebra. For $a \in E$, define $2a := a \oplus a$, if the sum exists, and by induction, $na := (n - 1)a \oplus a$, $n \in \mathrm{N}$, if the sum exists. With an appropriate definition of archimedean property in an effect algebra, and a subsequent variation of Wyler's proof, it can be shown that a ϕ-symmetric effect algebra is archimedean if and only if its universal group is.

Definition 3.3. A partially-ordered abelian group G is *archimedean* if, given $g, h \in G$, $ng \leq h$ for all $n \in \mathrm{N}$ implies $g \leq 0$. An effect algebra E is archimedean if na exists for all $n \in \mathrm{N}$ implies that $a = 0$.

If G is lattice-ordered, the archimedean condition is equivalent to the following condition (Proposition 1.23 in Goodearl, 1986): for $g, h \in G^+$, if $ng \leq h$ for all $n \in \mathrm{N}$, then $g = 0$.

Theorem 3.4. Let E be an effect algebra that has the ϕ-symmetry property. Then E is archimedean if and only if its universal group $\mathcal{G}$ is archimedean.

Proof. Assume $\mathcal{G}$ is archimedean. If, for $a \in E$, na exists for all $n \in \mathrm{N}$, then $a \leq 0$ in $\mathcal{G}$. Consequently, $a = 0$. Conversely, suppose E is archimedean. Let $[W_1], [W_2] \in \mathcal{G}^+$ such that $n[W_1] \leq [W_2]$ for all $n \in \mathrm{N}$. Since $\mathcal{G}$ is lattice-ordered, it is enough to show that $[W_1] = [0]$. We use induction on the length of W_2. If the length is 1, then the reader may easily verify using Scholium 2.10.1 of Ravindran (1996) and the archimedean property of E that $[W_1] = [0]$.

Suppose we have proved for all $[W_1] \in \mathcal{G}^+$ and all equivalence classes of words W_2 of length less than k that $n[W_1] \leq [W_2]$ for all $n \in \mathrm{N}$ implies $[W_1] = [0]$. Let W be a word of length k, say $W = (a_1, \ldots, a_k) = (a_1, \ldots, a_{k-1}) + (a_k)$ and assume $n[W_1] \leq [W] = [(a_1, \ldots, a_{k-1})] + [a_k]$ for all $n \in \mathrm{N}$. Then $n[W_1] - [a_k] \leq [(a_1, \ldots, a_{k-1})]$ for all $n \in \mathrm{N}$. Now, in a lattice-ordered group G, $l(g \vee 0) = lg \vee (l - 1)g \vee \ldots \vee g \vee 0$ for all $l \in \mathrm{N}$ (see, for example, Goodearl, 1986). Therefore, for all $m, n \in \mathrm{N}$, we have

$$
\begin{aligned}
& m((n[\mathrm{W}_1] - [a_k]) \vee [0]) \\
& \quad = m(n[W_1] - [a_k]) \vee (m - 1)(n[W_1] - [a_k]) \vee \cdots \\
& \qquad \vee n([W_1] - [a_k]) \vee 0 \\
& \quad = (mn[W_1] - m[a_k]) \vee ((m - 1)n[W_1] - (m - 1)[a_k]) \vee \cdots \\
& \qquad \vee (n[W_1] - n[a_k]) \vee 0 \\
& \quad \leq (mn[W_1] - [a_k]) \vee ((m - 1)n[W_1] - [a_k]) \vee \cdots \\
& \qquad \vee (n[W_1] - [a_k]) \vee 0 \\
& \quad \leq [(a_1, \ldots, a_{k-1})] \vee [(a_1, \ldots, a_{k-1})] \vee \cdots \vee [(a_1, \ldots, a_{k-1})] \vee 0 \\
& \quad = [(a_1, \ldots, a_{k-1})]
\end{aligned}
$$

where the second to last inequality follows from the fact that $0 \leq [a_k]$. Since $(n[W_1] - [a_k]) \vee [0] \in \mathcal{G}^+$, we can apply the induction hypothesis to conclude that $(n[W_1] - [a_k]) \vee [0] = 0$ for all $n \in \mathrm{N}$. Thus $n[W_1] - [a_k] \leq [0]$ for all $n \in \mathrm{N}$. It can now be argued as in the case of the induction step corresponding to $|W_2| = 1$ to arrive at the conclusion that $[W_1] = [0]$.

We conclude by proving the following theorem, which is the analog of Theorem 4.14 in Goodearl (1986) for the case of ϕ-symmetric effect algebras. Also see Belluce (1986) and Dvurečenskij (n.d.).

Theorem 3.5. Let E be an effect algebra that has the ϕ-symmetry property. Then the following are equivalent.

(i) $\mathcal{S}(E)$ is full.
(ii) E is isomorphic to $\hat{E}^{\mathcal{S}(}E)$.
(iii) E is archimedean.

Proof. (i) $\Rightarrow$ (ii): This follows from Proposition 3.2.

(ii) $\Rightarrow$ (iii): Suppose, for $a \in E$, na exists for all $n \in \mathrm{N}$. Then $n\hat{a} \leq \hat{u}$ for all $n \in \mathrm{N}$ by (ii). Thus, for each $s \in \mathcal{S}(E)$, we have $ns(a) \leqq 1$ in [0, 1]. Since the interval effect algebra [0, 1] is archimedean, we have $s(a) = 0$. This being true for all $s \in \mathcal{S}(\mathrm{E})$, we conclude that $\hat{a} = \hat{0}$; so, again by (ii), $a = 0$.

(iii) ⇒ (i): If E is archimedean, then its universal group $\mathcal{G}$ is archimedean by Theorem 3.4. By Theorem 4.14 of Goodearl (1986), $S(\mathcal{G}, u)$ is order-determining in $\mathcal{G}$. Hence $\mathcal{S}(E)$ is full.

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REFERENCES

Baer, R. (1949). Free sums of groups and their generalizations. An analysis of the associative law, *American Journal of Mathematics,* **71**, 706–742.

Belluce, L. P. (1986). Semisimple algebras of infinite valued logic and Bold Fuzzy set theory, *Canadian Journal of Mathematics,* **38**, 1356–1379.

Bennett, M. K., and Foulis, D. J. (1995). Phi-symmetric effect algebras, *Foundations of Physics,* **25**, 1699–1722.

Bennett, M. K., and Foulis, D. J. (n.d.) Interval and scale effect algebras, *Advances in Mathematics.*

Chang, C. C. (1958). Algebraic analysis of many valued logics, *Transactions of the American Mathematical Society,* **88**, 467–490.

Dvurečenskij, A. (n.d.) Fuzzy set representation of some quantum structures, *Fuzzy Set and Systems.*

Foulis, D. J., and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics,* **24**, 1325–1346.

Goodearl, K. R. (1986). *Partially Ordered Abelian Groups with Interpolation*, American Mathematical Society, Providence, Rhode Island.

Kôpka, F., and Chovanec, F. (1996). Boolean D-posets, *Tatra Mountains* **10.**

Hedlíková, J., and Pulmannov́, S. (1995). Generalized difference posets and orthoalgebras, preprint.

Ravindran, K. (1996). On a structure theory of effect algebras, Ph.D dissertation, Kansas State University, Manhattan, Kansas.

Wyler, O. (1966). Clans, *Compositio Mathematica,* **17**, 172–189.